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$Y(so(5))$ symmetry of the nonlinear Schrödinger model with four-components

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Abstract

The quantum nonlinear Schrödinger model with four-component fermions exhibits a $Y(so(5))$ symmetry when considered on an infinite interval. The constructed generators of the Yangian are proved to satisfy the Drinfel'd formula and, furthermore, the RTT relation with the general form of a rational R-matrix given by Yang–Baxterization associated with $so(5)$ algebraic structure.

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1. Introduction

Many one-dimensional nonlinear models possess $Y(sl(2))$ symmetry. The notable examples include the Hubbard model [1] and its extension [2], the Haldance–Shastry model [3–8], that are chain models, and, as explored by Murakami and Wadati [9], the nonlinear Schrödinger (NLS) model with spins. To our knowledge so far the demonstrated Yangian symmetry exhibited by physical models is only $Y(sl(2))$. However, it has been proposed by Zhang [10] that the antiferrromagnetic (AF) and superconducting (SC) phases of high-T_c cuprates show promise to be unified by an $so(5)$ symmetry principle [10]. Possible support for this proposal arose from numerical investigations in models for high-T_c materials. In particular, it was shown that the low-energy excitations can be classified in terms of an $so(5)$ symmetry multiplet structure [11, 12]. Subsequently, extended Hubbard models and a two-leg ladder model related to $so(5)$ symmetry have been introduced and analysed in detail [13–15]. While these models do not exhibit a higher symmetry, $Y(so(5))$ symmetry. In this paper we would like to show that the NLS model with four-component fermions will have a larger symmetry: the Yangian symmetry $Y(so(5))$, which is realized on a infinite interval. This paper can be viewed as an extension of [9] with mainly two different features: (1) the relations for $Y(so(5))$ are much more complicated than $Y(sl(2))$s, and (2) in comparison to the new realization of the Yangian by Drinfel'd [16], we should complete the calculations based on the RTT relation. This paper is organized as follows: in section 2 the commutativity between the NLS model with $so(5)$ symmetry and the continuous realization of $Y(so(5))$ is presented. In section 3 we recast the relations for $Y(so(5))$ given by Drinfel'd to the simplest form with the help of Jacobi identities. Then the presented operators $I_{ab}^{(1)}$ and $I_{ab}^{(2)}$ will be shown to satisfy $Y(so(5))$. In section 4 we start from the RTT relation and show that it yields the simplified form of $Y(so(5))$, that not only confirms our result, but also explicitly gives the 'new realization' of $Y(so(5))$ given in [16].

2. Hamiltonian and realization of $Y(so(5))$

In this section we will give an explicit expression for the genrators of $Y(so(5))$ commuting with the NLS model of four-component fermions.

The Hamiltonian for the NLS model of four-component fermions is given by the following expression:

$$H = \int \{\partial_x \psi_\alpha^+ \partial_x \psi_\alpha + c\psi_\beta^+ \psi_\alpha^+ \psi_\alpha \psi_\beta\}\, \mathrm{d}x. \tag{1}$$

Here $\psi_\alpha(x)$ ($\alpha = 1, 2, 3, 4$) denote fermionic field operators which satisfy canonical anticommutation relations

$$\begin{aligned} &[\psi_\alpha^+(x), \psi_\beta^+(y)]_+ = 0 \qquad [\psi_\alpha(x), \psi_\beta(y)]_+ = 0 \\ &[\psi_\alpha(x), \psi_\beta^+(y)]_+ = \delta_{\alpha\beta}\delta(x-y). \end{aligned} \tag{2}$$

We can construct Yangian generators as follows ($a, b = 1, 2, 3, 4, 5$):

$$\begin{aligned} &I_{ab} = I_{ab}^{(1)} = \int I_{ab}(x)\,\mathrm{d}x \qquad I_{ab}(x) = \tfrac{1}{2}\psi_\alpha^+(x)\Gamma_{\alpha\beta}^{ab}\psi_\beta(x) \\ &J_{ab} = I_{ab}^{(2)} = -\frac{\mathrm{i}}{2}\int \mathrm{d}x\, \psi_\alpha^+(x)\Gamma_{\alpha\beta}^{ab}\partial_x\psi_\beta(x) - \frac{\mathrm{i}c}{2}\iint \mathrm{d}x\,\mathrm{d}y\,\epsilon(x-y)I_{ac}(x)I_{cb}(y) \end{aligned} \tag{3}$$

where $\Gamma^{ab} = -\mathrm{i}\Gamma^a\Gamma^b$ and Γ^a are 4×4 Dirac matrices, $c = h$, in comparison with [10]. It can be checked that the set $\{I_{ab}^{(1)}, I_{ab}^{(2)}\}$ commutes with the Hamiltonian of the NLS model: $[H, I_{ab}^{(1)}] = [H, I_{ab}^{(2)}] = 0$ or $[H, Y(so(5))] = 0$, where the set $\{I_{ab}, J_{ab}\}$ satisfies $Y(so(5))$, i.e. $Y(so(5))$ serves as a symmetry of the NLS model. In momentum space, equation (3) can be expressed as

$$\begin{aligned} &J_{ab} = I_{ab}^{(2)} = \int kI_{ab}(k,k)\,\mathrm{d}k - \frac{c}{\pi}\iint \mathrm{d}k\,\mathrm{d}p\,\mathrm{d}q\, k^{-1}I_{ac}(k+p,p)I_{cb}(q,k+q) \\ &I_{ab} = I_{ab}^{(1)} = \int I_{ab}(k,k)\,\mathrm{d}k \qquad I_{ab}(k,k') = \tfrac{1}{2}\psi_\alpha^+(k)\Gamma_{\alpha\beta}^{ab}\psi_\beta(k'). \end{aligned} \tag{4}$$

For four-component fermions the field operator $\psi(k) = [c_\sigma(k), d_\sigma^+(-k)]^T$ ($\sigma = \uparrow, \downarrow$ and T means transport), then equation (1) is changed into the following form:

$$\begin{aligned} H = &\int \sum_{\sigma=\uparrow,\downarrow} \{(k^2+4c)c_\sigma^+(k)c_\sigma(k) + d_\sigma^+(k)d_\sigma(k) - 1\}\,\mathrm{d}k \\ &+\frac{c}{\pi}\iiint \mathrm{d}k\,\mathrm{d}k'\,\mathrm{d}q\,\Bigg\{c_\uparrow^+\left(k+\frac{q}{2}\right)c_\downarrow^+\left(-k+\frac{q}{2}\right)c_\downarrow\left(-k'+\frac{q}{2}\right)c_\uparrow\left(k'+\frac{q}{2}\right) \\ &-c_\uparrow^+\left(k+\frac{q}{2}\right)d_\uparrow^+\left(-k+\frac{q}{2}\right)d_\uparrow\left(-k'+\frac{q}{2}\right)c_\uparrow\left(k'+\frac{q}{2}\right) \\ &-c_\uparrow^+\left(k+\frac{q}{2}\right)d_\downarrow^+\left(-k+\frac{q}{2}\right)d_\downarrow\left(-k'+\frac{q}{2}\right)c_\uparrow\left(k'+\frac{q}{2}\right) \\ &-c_\downarrow^+\left(k+\frac{q}{2}\right)d_\uparrow^+\left(-k+\frac{q}{2}\right)d_\uparrow\left(-k'+\frac{q}{2}\right)c_\downarrow\left(k'+\frac{q}{2}\right) \\ &-c_\downarrow^+\left(k+\frac{q}{2}\right)d_\downarrow^+\left(-k+\frac{q}{2}\right)d_\downarrow^+\left(-k'+\frac{q}{2}\right)c_\downarrow\left(k'+\frac{q}{2}\right) \end{aligned}$$

$$+d_\uparrow\left(k+\frac{q}{2}\right)d_\downarrow\left(-k+\frac{q}{2}\right)d_\downarrow^+\left(-k'+\frac{q}{2}\right)d_\uparrow^+\left(k'+\frac{q}{2}\right)\Bigg\}. \tag{5}$$

Since $c_\sigma(k)$ and $d_\sigma(k)$ denote different fermions, equation (5) gives the pair interaction. Equation (4) can be expressed in terms of Cartan–Weyl basis forms with

$$\begin{array}{lll} E_3^{(1)}=\int E_3(k,k)\,\mathrm{d}k & E_\pm^{(1)}=\int E_\pm(k,k)\,\mathrm{d}k & F_3^{(1)}=\int F_3(k,k)\,\mathrm{d}k \\ F_\pm^{(1)}=\int F_\pm(k,k)\,\mathrm{d}k & U_\pm^{(1)}=\int U_\pm(k,k)\,\mathrm{d}k & V_\pm^{(1)}=\int V_\pm(k,k)\,\mathrm{d}k \end{array} \tag{6}$$

$$\begin{aligned}
E_3^{(2)} &= \int kE_3(k,k)\,\mathrm{d}k - \frac{c}{\pi}\iiint \mathrm{d}k\,\mathrm{d}p\,\mathrm{d}q\,k^{-1}\{U_+(k+p,p)U_-(q,k+q) \\
&\qquad +V_+(k+p,p)V_-(q,k+q)+E_+(k+p,p)E_-(q,k+q)\} \\
E_\pm^{(2)} &= \int kE_\pm(k,k)\,\mathrm{d}k \mp \frac{c}{\pi}\iiint \mathrm{d}k\,\mathrm{d}p\,\mathrm{d}q\,k^{-1}\{U_\pm(k+p,p)F_\pm(q,k+q) \\
&\qquad -V_\pm(k+p)F_\mp(q,k+q)+E_3(k+p,p)E_\pm(q,k+q)\} \\
F_3^{(2)} &= \int kF_3(k,k)\,\mathrm{d}k - \frac{c}{\pi}\iiint \mathrm{d}k\,\mathrm{d}p\,\mathrm{d}q\,k^{-1}\{-U_+(k+p,p)U_-(q,k+q) \\
&\qquad +V_+(k+p,p)V_-(q,k+q)+F_+(k+p,p)F_-(q,k+q)\} \\
F_\pm^{(2)} &= \int kF_\pm(k,k)\,\mathrm{d}k \mp \frac{c}{\pi}\iiint \mathrm{d}k\,\mathrm{d}p\,\mathrm{d}q\,k^{-1}\{F_3(k+p,p)F_\pm(q,k+q) \\
&\qquad +V_\pm(k+p,p)E_\mp(q,k+q)-E_\pm(k+p,p)U_\mp(q,k+q)\} \\
U_\pm^{(2)} &= \int kU_\pm(k,k)\,\mathrm{d}k \mp \frac{c}{\pi}\iiint \mathrm{d}k\,\mathrm{d}p\,\mathrm{d}q\,k^{-1}\{E_\pm(k+p,p)F_\mp(q,k+q) \\
&\qquad +U_\pm(k+p,p)(F_3(q,k+q)-E_3(q,k+q))\} \\
V_\pm^{(2)} &= \int kV_\pm(k,k)\,\mathrm{d}k \pm \frac{c}{\pi}\iiint \mathrm{d}k\,\mathrm{d}p\,\mathrm{d}q\,k^{-1}\{E_\pm(k+p,p)F_\pm(q,k+q) \\
&\qquad -(E_3(k+p,p)+F_3(k+p,p))V_\pm(q,k+q)\}
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
E_+(k,k') &= \frac{1}{\sqrt{2}}(c_\uparrow^+(k)c_\downarrow(k') - d_\downarrow(-k)d_\uparrow(-k')^+) \\
E_-(k,k') &= \frac{1}{\sqrt{2}}(c_\downarrow^+(k)c_\uparrow(k') - d_\uparrow(-k)d_\downarrow(-k')^+) \\
F_+(k,k') &= \frac{-1}{\sqrt{2}}(d_\uparrow(-k)c_\downarrow(k') + d_\downarrow(-k)c_\uparrow(k')) \\
F_-(k,k') &= \frac{-1}{\sqrt{2}}(c_\uparrow^+(k)d_\downarrow^+(-k') + c_\downarrow^+(k)d_\uparrow^+(-k')) \\
U_+(k,k') &= -c_\uparrow^+(k)d_\uparrow^+(-k') \qquad U_-(k,k') = -d_\uparrow(-k)c_\uparrow(k') \\
V_+(k,k') &= -d_\downarrow(-k)c_\downarrow(k') \qquad V_-(k,k') = -c_\downarrow^+(k)d_\downarrow^+(-k') \\
E_3(k,k') &= \tfrac{1}{2}(c_\uparrow^+(k)c_\downarrow(k') - c_\downarrow^+(k)c_\downarrow(k') - d_\uparrow(-k)d_\uparrow^+(-k') + d_\downarrow(-k)d_\downarrow^+(-k')) \\
F_3(k,k') &= \tfrac{1}{2}(-c_\uparrow^+(k)c_\downarrow(k') - c_\downarrow^+(k)c_\downarrow(k') + d_\uparrow(-k)d_\uparrow^+(-k') + d_\downarrow(-k)d_\downarrow^+(-k')).
\end{aligned}$$

The relations for $U_\alpha^{(n)}$, $V_\alpha^{(n)}$, $E_\alpha^{(n)}$, $F_\alpha^{(n)}$ ($n = 1, 2$ and $\alpha = \pm, 3$) and $I_{ab}^{(1)}$, $I_{ab}^{(2)}$ are given by

$$\begin{aligned} E_3^{(n)} &= \frac{1}{h^{n-1}} I_{23}^{(n)} \qquad E_\pm^{(n)} = \frac{1}{\sqrt{2}h^{n-1}}(I_{34}^{(n)} \pm \mathrm{i} I_{42}^{(n)}) \qquad F_\pm^{(n)} = \frac{1}{\sqrt{2}h^{n-1}}(I_{45}^{(n)} \pm \mathrm{i} I_{14}^{(n)}) \\ F_3^{(n)} &= \frac{1}{h^{n-1}} I_{15}^{(n)} \qquad U_\pm^{(n)} = \frac{1}{2h^{n-1}}\{(I_{31}^{(n)} \pm \mathrm{i} I_{12}^{(n)}) - (I_{25}^{(n)} \pm \mathrm{i} I_{35}^{(n)})\} \\ V_\pm^{(n)} &= \frac{1}{2h^{n-1}}\{(I_{31}^{(n)} \pm \mathrm{i} I_{12}^{(n)}) + (I_{25}^{(n)} \pm \mathrm{i} I_{35}^{(n)})\}. \end{aligned} \tag{8}$$

Therefore, we have expressed the generators of $Y(so(5))$ in terms of fermions including their pairs. Equipped with the above relations we shall show that the I_{ab} and J_{ab} given by equation (3) really satisfy $Y(so(5))$.

3. Simplification of the commutation relations of $Y(so(5))$

The original commutation relations of the Yangian $Y(\boldsymbol{a})$ were given by Drinfel'd [17] in the form

$$[I_\lambda, I_\mu] = c_{\lambda\mu\nu} I_\nu \qquad [I_\lambda, J_\mu] = c_{\lambda\mu\nu} J_\nu \tag{9}$$

$$[J_\lambda, [J_\mu, I_\nu]] - [I_\lambda, [J_\mu, J_\nu]] = h^2 a_{\lambda\mu\nu\alpha\beta\gamma}\{I_\alpha, I_\beta, I_\gamma\} \tag{10}$$

where equation (10) does not work only for $sl(2)$, and

$$[[J_\lambda, J_\mu], [I_\sigma, J_\tau]] + [[J_\sigma, J_\tau], [I_\lambda, J_\mu]] = h^2(a_{\lambda\mu\nu\alpha\beta\gamma} c_{\sigma\tau\nu} + a_{\sigma\tau\nu\alpha\beta\gamma} c_{\lambda\mu\nu})\{I_\alpha, I_\beta, J_\gamma\} \tag{11}$$

where $c_{\lambda\mu\nu}$ are structure constants of a simple Lie algebra $\boldsymbol{a}$, h is a constant and

$$\begin{aligned} a_{\lambda\mu\nu\alpha\beta\gamma} &= \frac{1}{4!} c_{\lambda\alpha\sigma} c_{\mu\beta\tau} c_{\nu\gamma\rho} c_{\sigma\tau\rho} \\ \{x_1, x_2, x_3\} &= \sum_{i\neq j\neq k} x_i x_j x_k. \end{aligned} \tag{12}$$

Through the 'new realization of the Yangian' given in [16] by Drinfel'd the isomorphism between $Y(\boldsymbol{a})$ and $T_{ab}^{(n)}$ was established, where $T_{ab}(u)$ are matrix elements of the transfer matrix $T(u)$ obeying the RTT relation

$$\check{R}(u - v)(T(u) \otimes T(v)) = (T(v) \otimes T(u))\check{R}(u - v) \tag{13}$$

where $\check{R}(u)$ is the rational solution of the YBE. The expansion of $T(u)$ is made by

$$T_{ab}(u) = \delta_{ab} + \sum_{n=1}^{\infty} u^{-n} T_{ab}^{(n)}. \tag{14}$$

From the point of view of mathematics [16–18] together with the representation theory of $Y(sl(2))$ by Chari and Pressley [19] we have solved the basic problems of the Yangian associated with $sl(2)$ ($Y(sl(2))$). For $sl(2)$ equation (11) can be simplified to only one relation as was shown in [8]. Now we can show that besides equation (9) not all of the relations in equations (10) and (11) are independent. After tedious calculation we can prove that there is only one independent relation:

$$[I_{23}^{(2)}, I_{15}^{(2)}] = \frac{\mathrm{i}}{4!} h^2(\{I_{13}^{(1)}, I_{42}^{(1)}, I_{45}^{(1)}\} + \{I_{12}^{(1)}, I_{45}^{(1)}, I_{34}^{(1)}\} - \{I_{14}^{(1)}, I_{42}^{(1)}, I_{35}^{(1)}\} - \{I_{14}^{(1)}, I_{34}^{(1)}, I_{25}^{(1)}\}) \tag{15}$$

where $I_{23}^{(1)}$ and $I_{15}^{(1)}$ are the Cartan subset. All the other relations in equations (10) and (11) can be generated on the basis of equation (15) by using Jacobi identities together with equation (9).

Therefore, for $Y(so(5))$, equations (9)–(11) can also be expressed with equations (16) and (17)

$$[I_{ab}^{(1)}, I_{cd}^{(1)}] = \mathrm{i}(\delta_{bc} I_{ad}^{(1)} + \delta_{ad} I_{bc}^{(1)} - \delta_{ac} I_{bd}^{(1)} - \delta_{bd} I_{ac}^{(1)}) \tag{16}$$

$$\begin{aligned} &[I_{ab}^{(1)}, I_{cd}^{(2)}] = \mathrm{i}(\delta_{bc} I_{ad}^{(2)} + \delta_{ad} I_{bc}^{(2)} - \delta_{ac} I_{bd}^{(2)} - \delta_{bd} I_{ac}^{(2)}) \\ &I_{ab}^{(1)} = -I_{ba}^{(1)} \qquad I_{ab}^{(2)} = -I_{ba}^{(2)} \qquad (a, b, c, d = 1, 2, 3, 4, 5) \end{aligned} \tag{17}$$

together with equation (15). The proof is direct: substituting the structure constants for $so(5)$ into equations (10) and (11), then carefully using Jacobi identities together with equation (9) and checking one by one the relations, finally based on equation (9) we generate all the relations given by equations (10) and (11) in terms of equation (15) and Jacobi identities. This process looks at the extension for $Y(sl(2))$ as discussed in [20]. A similar situation also occurs in $Y(su(3))$. A direct check verifies that equation (3) does indeed satisfy equations (15)–(17).

4. RTT relations and $Y(so(5))$

We have proved that the operators I_{ab} and J_{ab} shown by equation (3) satisfy equations (15)–(17) and hence equations (9)–(11). To independently check this statement and furthermore show the consistency between equation (15) and the 'new realization' of the Yangian given by Drinfel'd [16], we have to solve the RTT relation for $so(5)$, i.e. starting from equations (13) and (14) we shall give the general forms of $T_{ab}^{(n)}$ and verify that when $n = 1, 2$ they will give equations (15)–(17). With this goal we should begin with the rational R-matrix associated with $so(5)$. For $so(N)$ the general form has been given in [21] through Yang–Baxterization [22]: $\check{R}(u) = f(u)\{u^2 P + u(q_1 P + xA + q_2 I)x + qI\}$, where $q_1 = (1 - \frac{N}{2})x$, $q_2 = -x$, $q = q_1 q_2$, x is a constant and the elements of permutation matrix P, unit matrix I and block-diagonal matrix A are $P_{cd}^{ab} = \delta_{ad}\delta_{bc}$, $I_{cd}^{ab} = \delta_{ac}\delta_{bd}$ and $A_{cd}^{ab} = \delta_{a,-b}\delta_{c,-d}$ when $a, b, c, d \in \{\frac{(-N+1)}{2}, \frac{(-N+3)}{2}, \ldots, \frac{(N-3)}{2}, \frac{(N-1)}{2}\}$. For $so(5)$, $N = 5$, the $\check{R}$-matrix is changed into the following form:

$$\check{R}(u) = f(u)\{u^2 P + u(A - I - \tfrac{3}{2}P)x + \tfrac{3}{2}x^2 I\}. \tag{18}$$

For our discussion we restrict ourselves to working in five-dimensional auxiliary space, whereas each element T_{ab} is a quantum operator, where $a, b = 0, \pm 1, \pm 2$. In comparison with Drinfel'd [16], we express the transfer matrix $T(u)$ as

$$T_{ab}(u) = \delta_{ab} + \sum_{n=1}^{\infty} \left(\frac{x}{u}\right)^n T_{ab}^{(n)}. \tag{19}$$

Substituting equations (18) and (19) into equation (13), by direct calculation, we obtain the following relations:

$$\begin{aligned} &[T_{bc}^{(n+2)}, T_{ad}^{(m)}] - 2[T_{bc}^{(n+1)}, T_{ad}^{(m+1)}] + [T_{bc}^{(n)}, T_{ad}^{(m+2)}] + (T_{ic}^{(n+1)} T_{-id}^{(m)} \\ &\quad - T_{ic}^{(n)} T_{-id}^{(m+1)})\delta_{a,-b} + (T_{ai}^{(m+1)} T_{b-i}^{(n)} - T_{ai}^{(m)} T_{b-i}^{(n+1)})\delta_{c,-d} - \tfrac{3}{2}([T_{bc}^{(n+1)}, T_{ad}^{(m)}] \\ &\quad - [T_{bc}^{(n)}, T_{ad}^{(m+1)}]) - T_{ac}^{(n+1)} T_{bd}^{(m)} + T_{ac}^{(n)} T_{bd}^{(m+1)} + T_{ac}^{(m)} T_{bd}^{(n+1)} \\ &\quad - T_{ac}^{(m+1)} T_{bd}^{(n)} + \tfrac{3}{2}(T_{ac}^{(n)} T_{bd}^{(m)} - T_{ac}^{(m)} T_{bd}^{(n)}) = 0 \end{aligned} \tag{20}$$

$$[T_{bc}^{(1)}, T_{ad}^{(m)}] + T_{-cd}^{(m)}\delta_{a,-b} - T_{a-b}^{(m)}\delta_{c,-d} - T_{bd}^{(m)}\delta_{ac} + T_{ac}^{(m)}\delta_{bd} = 0 \tag{21}$$

$$[T_{bc}^{(n)}, T_{ad}^{(1)}] + T_{b-a}^{(n)}\delta_{c,-d} - T_{-dc}^{(n)}\delta_{a,-b} - T_{bd}^{(n)}\delta_{ac} + T_{ac}^{(n)}\delta_{bd} = 0. \tag{22}$$

From equation (21) the exchange ($a \leftrightarrow b, c \leftrightarrow d, n \leftrightarrow m$) leads to equation (22). This indicates that equation (21) is exactly the same as equation (22). For the convenience of the following calculation, we define $T_{ab}^{(n)}$ as follows:

$$
\begin{array}{lll}
T_{22}^{(n)} - T_{-2-2}^{(n)} = 2E_3^{(n)} & T_{22}^{(n)} + T_{-2-2}^{(n)} = 2\tilde{E}_3^{(n)} & T_{21}^{(n)} - T_{-1-2}^{(n)} = 2U_+^{(n)} \\
T_{21}^{(n)} + T_{-1-2}^{(n)} = 2\tilde{U}_+^{(n)} & T_{20}^{(n)} - T_{0-2}^{(n)} = 2E_+^{(n)} & T_{20}^{(n)} + T_{0-2}^{(n)} = 2\tilde{E}_+^{(n)} \\
T_{2-1}^{(n)} - T_{1-2}^{(n)} = 2V_+^{(n)} & T_{2-1}^{(n)} + T_{1-2}^{(n)} = 2\tilde{V}_+^{(n)} & T_{12}^{(n)} - T_{-2-1}^{(n)} = 2U_-^{(n)} \\
T_{12}^{(n)} + T_{-2-1}^{(n)} = 2\tilde{U}_-^{(n)} & T_{11}^{(n)} - T_{-1-1}^{(n)} = 2F_3^{(n)} & T_{11}^{(n)} + T_{-1-1}^{(n)} = 2\tilde{F}_3^{(n)} \\
T_{10}^{(n)} - T_{0-1}^{(n)} = 2F_+^{(n)} & T_{10}^{(n)} + T_{0-1}^{(n)} = 2\tilde{F}_+^{(n)} & T_{02}^{(n)} - T_{-20}^{(n)} = 2E_-^{(n)} \\
T_{02}^{(n)} + T_{-20}^{(n)} = 2\tilde{E}_-^{(n)} & T_{01}^{(n)} - T_{-10}^{(n)} = 2F_-^{(n)} & T_{01}^{(n)} + T_{-10}^{(n)} = 2\tilde{F}_-^{(n)} \\
T_{1-1}^{(n)} = Y_+^{(n)} \quad T_{-11}^{(n)} = Y_-^{(n)} & T_{-22}^{(n)} = X_-^{(n)} & T_{2-2}^{(n)} = X_+^{(n)} \\
T_{-12}^{(n)} - T_{-21}^{(n)} = 2V_-^{(n)} & T_{-12}^{(n)} + T_{-21}^{(n)} = 2\tilde{V}_-^{(n)} & T_{00}^{(n)} = I_0^{(n)}.
\end{array}
\tag{23}
$$

This definition of $T_{ab}^{(n)}$ is expressed in terms of matrix form with

$$
T_{ab}^{(n)} = \begin{pmatrix}
\tilde{E}_3^{(n)} + E_3^{(n)} & \tilde{U}_+^{(n)} + U_+^{(n)} & \tilde{E}_+^{(n)} + E_+^{(n)} & \tilde{V}_+^{(n)} + V_+^{(n)} & X_+^{(n)} \\
\tilde{U}_-^{(n)} + U_-^{(n)} & \tilde{F}_3^{(n)} + F_3^{(n)} & \tilde{F}_+^{(n)} + F_+^{(n)} & Y_+^{(n)} & \tilde{V}_+^{(n)} - V_+^{(n)} \\
\tilde{E}_-^{(n)} + E_-^{(n)} & \tilde{F}_-^{(n)} + F_-^{(n)} & I_0^{(n)} & \tilde{F}_+^{(n)} - F_+^{(n)} & \tilde{E}_+^{(n)} - E_+^{(n)} \\
\tilde{V}_-^{(n)} + V_-^{(n)} & Y_-^{(n)} & \tilde{F}_-^{(n)} - F_-^{(n)} & \tilde{F}_3^{(n)} - F_3^{(n)} & \tilde{U}_+^{(n)} - U_+^{(n)} \\
X_-^{(n)} & \tilde{V}_-^{(n)} - V_-^{(n)} & \tilde{E}_-^{(n)} - E_-^{(n)} & \tilde{U}_-^{(n)} - U_-^{(n)} & \tilde{E}_3^{(n)} - E_3^{(n)}
\end{pmatrix}.
\tag{24}
$$

For equation (22), when $n = 1$, substituting equations (23) into (22), by extensive calculation, we can obtain the algebraic relations in the appendix equations (30) and the following constraint conditions:

$$
X_\pm^{(1)} = Y^{(1)}\pm = \tilde{U}_\pm^{(1)} = \tilde{V}_\pm^{(1)} = \tilde{E}_\pm^{(1)} = \tilde{F}_\pm^{(1)} = 0 \qquad \tilde{E}_3^{(1)} = \tilde{F}_3^{(1)} = I_0^{(1)}.
\tag{25}
$$

Therefore, $T^{(1)}$ is expressed in terms of the generators of the Lie algebra $so(5)$ with

$$
T_{ab}^{(1)} = \begin{pmatrix}
I_0^{(1)} + E_3^{(1)} & U_+^{(1)} & E_+^{(1)} & V_+^{(1)} & 0 \\
U_-^{(1)} & I_0^{(1)} + F_3^{(1)} & F_+^{(1)} & 0 & -V_+^{(1)} \\
E_-^{(1)} & F_-^{(1)} & I_0^{(n)} & -F_+^{(1)} & -E_+^{(1)} \\
V_-^{(1)} & 0 & -F_-^{(1)} & I_0^{(1)} - F_3^{(1)} & -U_+^{(1)} \\
0 & -V_-^{(1)} & -E_-^{(1)} & -U_-^{(1)} & I_0^{(1)} - E_3^{(1)}
\end{pmatrix}
\tag{26}
$$

where $I_0^{(1)}$ is a Casimor operator. While for arbitrary n, substituting equations (23) into (22) we can calculate commutators between $T^{(1)}$ and $T^{(n)}$ in appendix equations (33) and (34).

When taking $n = 1$, substituting equations (23) into (20) by direct calculations, we can obtain the following constraint conditions:

$$
\begin{aligned}
&\tilde{E}_3^{(2)} - I_0^{(2)} = \tfrac{1}{4}([U_+^{(1)}, U_-^{(1)}]_+ + [V_+^{(1)}, V_-^{(1)}]_+ - 2[F_+^{(1)}, F_-^{(1)}]_+ - [E_+^{(1)}, E_-^{(1)}]_+ + 2{E_3^{(1)}}^2) \\
&\tilde{F}_3^{(2)} - I_0^{(2)} = \tfrac{1}{4}([U_+^{(1)}, U_-^{(1)}]_+ + [V_+^{(1)}, V_-^{(1)}]_+ - 2[E_+^{(1)}, E_-^{(1)}]_+ - [F_+^{(1)}, F_-^{(1)}]_+ + 2{F_3^{(1)}}^2) \\
&\tilde{U}_\pm^{(2)} = \tfrac{1}{4}([E_\pm^{(1)}, F_\mp^{(1)}]_+ + [E_3^{(1)}, U_\pm^{(1)}]_+ + [F_3^{(1)}, U_\pm^{(1)}]_+) \\
&\tilde{V}_\pm^{(2)} = \tfrac{1}{4}(-[E_\pm^{(1)}, F_\pm^{(1)}]_+ + [E_3^{(1)}, V_\pm^{(1)}]_+ - [F_3^{(1)}, V_\pm^{(1)}]_+) \\
&\tilde{E}_\pm^{(2)} = \tfrac{1}{4}([E_3^{(1)}, E_\pm^{(1)}]_+ - [V_\pm^{(1)}, F_\mp^{(1)}]_+ + [F_\pm^{(1)}, U_\pm^{(1)}]_+) \\
&\tilde{F}_\pm^{(2)} = \tfrac{1}{4}([E_\mp^{(1)}, V_\pm^{(1)}]_+ + [E_\pm^{(1)}, U_\mp^{(1)}]_+ + [F_3^{(1)}, F_\pm^{(1)}]_+) \\
&X_\pm^{(2)} = -\tfrac{1}{4}([E_\pm^{(1)}, E_\pm^{(1)}]_+ + 2[U_\pm^{(1)}, V_\pm^{(1)}]_+) \\
&Y_\pm^{(2)} = -\tfrac{1}{4}([F_\pm^{(1)}, F_\pm^{(1)}]_+ - 2[U_\mp^{(1)}, V_\pm^{(1)}]_+).
\end{aligned}
\tag{27}
$$

Substituting equations (23) into (20), by using the constraint conditions equations (27), equations (25) we can reduce the following independent relations:

$$
\begin{aligned}
&[E_3^{(n)}, F_3^{(2)}] + \tfrac{1}{4}([U_+^{(1)}, \tilde{U}_-^{(n)}]_+ - [U_-^{(1)}, \tilde{U}_+^{(n)}]_+ + [V_-^{(1)}, \tilde{V}_+^{(n)}]_+ - [V_+^{(1)}, \tilde{V}_-^{(n)}]_+) = 0\\
&[E_3^{(n)}, F_3^{(2)}] = [E_3^{(2)}, F_3^{(n)}]\\
&[E_3^{(n)}, E_3^{(2)}] + \tfrac{1}{4}([U_-^{(1)}, \tilde{U}_+^{(n)}]_+ - [U_+^{(1)}, \tilde{U}_-^{(n)}]_+ + [V_-^{(1)}, \tilde{V}_+^{(n)}]_+\\
&\qquad - [V_+^{(1)}, \tilde{V}_-^{(n)}]_+ + [E_-^{(1)}, \tilde{E}_+^{(n)}]_+ - [E_+^{(1)}, \tilde{E}_-^{(n)}]_+) = 0\\
&[F_3^{(n)}, F_3^{(2)}] + \tfrac{1}{4}([U_+^{(1)}, \tilde{U}_-^{(n)}]_+ - [U_-^{(1)}, \tilde{U}_+^{(n)}]_+ + [V_+^{(1)}, \tilde{V}_-^{(n)}]_+\\
&\qquad - [V_-^{(1)}, \tilde{V}_+^{(n)}]_+ + [F_-^{(1)}, \tilde{F}_+^{(n)}]_+ - [F_+^{(1)}, \tilde{F}_-^{(n)}]_+) = 0\\
&[I_0^{(n)}, I_0^{(2)}] + ([E_-^{(1)}, \tilde{E}_+^{(n)}]_+ - [E_+^{(1)}, \tilde{E}_-^{(n)}]_+ + [F_-^{(1)}, \tilde{F}_+^{(n)}]_+ - [F_+^{(1)}, \tilde{F}_-^{(n)}]_+) = 0\\
&[E_3^{(n)}, I_0^{(2)}] + \tfrac{1}{2}([E_+^{(1)}, E_-^{(n)}]_+ - [E_+^{(n)}, E_-^{(1)}]_+) = 0 \qquad (28)\\
&[F_3^{(n)}, I_0^{(2)}] + \tfrac{1}{2}([F_+^{(1)}, F_-^{(n)}]_+ - [F_-^{(1)}, F_+^{(n)}]_+) = 0\\
&[\tilde{E}_3^{(n)}, F_3^{(2)}] + \tfrac{1}{4}([U_+^{(1)}, U_-^{(n)}]_+ - [U_-^{(1)}, U_+^{(n)}]_+ + [V_-^{(1)}, V_+^{(n)}]_+ - [V_+^{(1)}, V_-^{(n)}]_+) = 0\\
&[\tilde{F}_3^{(n)}, E_3^{(2)}] + \tfrac{1}{4}([U_+^{(n)}, U_-^{(1)}]_+ - [U_-^{(n)}, U_+^{(1)}]_+ + [V_-^{(1)}, V_+^{(n)}]_+ - [V_+^{(1)}, V_-^{(n)}]_+) = 0\\
&[\tilde{E}_3^{(n)}, E_3^{(2)}] + \tfrac{1}{4}([U_-^{(1)}, U_+^{(n)}]_+ - [U_+^{(1)}, U_-^{(n)}]_+ + [V_-^{(1)}, V_+^{(n)}]_+\\
&\qquad - [V_+^{(1)}, V_-^{(n)}]_+ + [E_-^{(1)}, E_+^{(n)}]_+ - [E_+^{(1)}, E_-^{(n)}]_+) = 0\\
&[\tilde{F}_3^{(n)}, F_3^{(2)}] + \tfrac{1}{4}([U_+^{(1)}, U_-^{(n)}]_+ - [U_-^{(1)}, U_+^{(n)}]_+ + [V_-^{(1)}, V_+^{(n)}]_+\\
&\qquad - [V_+^{(1)}, V_-^{(n)}]_+ + [F_-^{(1)}, F_+^{(n)}]_+ - [F_+^{(1)}, F_-^{(n)}]_+) = 0
\end{aligned}
$$

$$
\begin{aligned}
E_3^{(n+1)} &= [E_+^{(n)}, E_-^{(2)}] + \tfrac{1}{4}([I_0^{(n)}, E_3^{(1)}]_+ - 2[I_0^{(1)}, E_3^{(n)}]_+ - [E_-^{(1)}, \tilde{E}_+^{(n)}]_+\\
&\quad - [F_-^{(1)}, \tilde{F}_+^{(n)}]_+ + [F_+^{(1)}, \tilde{F}_-^{(n)}]_+ + [U_-^{(1)}, \tilde{U}_+^{(n)}]_+ + [V_-^{(1)}, \tilde{V}_+^{(n)}]_+ + [E_3^{(1)}, \tilde{E}_3^{(n)}]_+)\\
F_3^{(n+1)} &= [F_+^{(n)}, F_-^{(2)}] + \tfrac{1}{4}([I_0^{(n)}, F_3^{(1)}]_+ - 2[I_0^{(1)}, F_3^{(n)}]_+ - [F_-^{(1)}, \tilde{F}_+^{(n)}]_+\\
&\quad - [E_-^{(1)}, \tilde{E}_+^{(n)}]_+ + [E_+^{(1)}, \tilde{E}_-^{(n)}]_+ + [U_+^{(1)}, \tilde{U}_-^{(n)}]_+\\
&\quad - [V_-^{(1)}, \tilde{V}_+^{(n)}]_+ + [F_3^{(1)}, \tilde{F}_3^{(n)}]_+)\\
\tilde{E}_3^{(n+1)} - I_0^{(n+1)} &= [\tilde{E}_+^{(n)}, E_-^{(2)}] + \tfrac{1}{4}(2[I_0^{(n)}, I_0^{(1)}]_+ - 2[I_0^{(1)}, \tilde{E}_3^{(n)}]_+ \qquad (29)\\
&\quad - [E_-^{(1)}, E_+^{(n)}]_+ - [F_-^{(1)}, F_+^{(n)}]_+ - [F_+^{(1)}, F_-^{(n)}]_+ + [U_-^{(1)}, U_+^{(n)}]_+\\
&\quad + [V_-^{(1)}, V_+^{(n)}]_+ + [E_3^{(1)}, E_3^{(n)}]_+)\\
\tilde{F}_3^{(n+1)} - I_0^{(n+1)} &= [\tilde{F}_+^{(n)}, F_-^{(2)}] + \tfrac{1}{4}(2[I_0^{(n)}, I_0^{(1)}]_+ - 2[I_0^{(1)}, \tilde{F}_3^{(n)}]_+\\
&\quad - [F_-^{(1)}, F_+^{(n)}]_+ - [E_-^{(1)}, E_+^{(n)}]_+ - [E_+^{(1)}, E_-^{(n)}]_+ + [U_+^{(1)}, U_-^{(n)}]_+\\
&\quad + [V_-^{(1)}, V_+^{(n)}]_+ + [F_3^{(1)}, F_3^{(n)}]_+)
\end{aligned}
$$

where equations (28) are the constraint conditions and equations (29) are the iterative relation. All the other relations in equation (20) can be generated on the basis of equations (28) and (29) by making use of Jacobi indentities together with equations (30), (31) and constraint condition equations (28). Taking $n = 2$ in equations (28) and (29), we derive only one independent relation, which corresponds to equation (15) in the appendix equation (32). It is emphasized that because of the iterative relation equations (29), only $T^{(1)}$ and $T^{(2)}$ are basic ones. To satisfy all the relations with $T^{(n)} (n \geqslant 3)$ it is enough to look for the constraints on $T^{(3)}$ that in turn provide the constraints on $T^{(2)}$ itself. In brief, we have verified the equivalence between equations (15)–(17) and the expansions of the RTT relations for $m = 0, n \leqslant 2$ for $Y(so(5))$.

5. Conclusion

In this paper we have simplified the relations satisfied by $Y(so(5))$ to equations (15)–(17) and shown their correspondence to the RTT relation. Also we have made the realization of

$Y(so(5))$ in the NLS model of four-component fermions and shown Yangian symmetry for such an NLS model.

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Appendix

In the Cartan–Weyl basis equation (8), equations (16) and (17) are changed into the following forms:

$$\begin{array}{lll}
[E_3^{(1)}, U_\pm^{(1)}] = \pm U_\pm^{(1)} & [E_3^{(1)}, E_\pm^{(1)}] = \pm E_\pm^{(1)} & \\
[E_3^{(1)}, V_\pm^{(1)}] = \pm V_\pm^{(1)} & [E_3^{(1)}, F_\alpha^{(1)}] = 0 & [E_\pm^{(1)}, U_\mp^{(1)}] = \mp F_\pm^{(1)} \\
[E_\pm^{(1)}, U_\pm^{(1)}] = 0 & [U_\pm^{(1)}, V_\pm^{(1)}] = 0 & [V_+^{(1)}, V_-^{(1)}] = E_3^{(1)} + F_3^{(1)} \\
[U_\pm^{(1)}, V_\mp^{(1)}] = 0 & [E_\pm^{(1)}, V_\pm^{(1)}] = 0 & [E_\pm^{(1)}, V_\mp^{(1)}] = \pm F_\mp^{(1)} \\
[U_+^{(1)}, U_-^{(1)}] = E_3^{(1)} - F_3^{(1)} & [E_\pm^{(1)}, F_\pm^{(1)}] = \mp V_\pm^{(1)} & \\
[E_\pm^{(1)}, F_\mp^{(1)}] = \pm U_\pm^{(1)} & [F_\pm^{(1)}, V_\mp^{(1)}] = \mp E_\mp^{(1)} & [F_\pm^{(1)}, V_\pm^{(1)}] = 0 \\
[F_\pm^{(1)}, U_\pm^{(1)}] = \mp E_\pm^{(1)} & [F_\pm^{(1)}, U_\mp^{(1)}] = 0 & \\
[F_3^{(1)}, U_\pm^{(1)}] = \mp U_\pm^{(1)} & [F_3^{(1)}, V_\pm^{(1)}] = \pm V_\pm^{(1)} & \\
[F_3^{(1)}, F_\pm^{(1)}] = \pm F_\pm^{(1)} & [F_3^{(1)}, E_\pm^{(1)}] = 0 & \\
[E_+^{(1)}, E_-^{(1)}] = E_3^{(1)} & [F_+^{(1)}, F_-^{(1)}] = F_3^{(1)} &
\end{array} \tag{30}$$

$$\begin{array}{ll}
[E_3^{(2)}, U_\pm^{(1)}] = \pm U_\pm^{(2)} = [E_3^{(1)}, U_\pm^{(2)}] & [E_3^{(2)}, E_\pm^{(1)}] = \pm E_\pm^{(2)} = [E_3^{(1)}, E_\pm^{(2)}] \\
[E_3^{(2)}, V_\pm^{(1)}] = \pm V_\pm^{(2)} = [E_3^{(1)}, V_\pm^{(2)}] & [E_3^{(2)}, F_\alpha^{(1)}] = 0 = [E_3^{(1)}, F_\alpha^{(2)}] \\
[E_\pm^{(2)}, U_\mp^{(1)}] = \mp F_\pm^{(2)} = [E_\pm^{(1)}, U_\mp^{(2)}] & [E_\pm^{(2)}, U_\pm^{(1)}] = 0 = [E_\pm^{(1)}, U_\pm^{(2)}] \\
[U_\pm^{(2)}, V_\pm^{(1)}] = 0 = [U_\pm^{(1)}, V_\pm^{(2)}] & [U_\pm^{(2)}, V_\mp^{(1)}] = 0 = [U_\pm^{(1)}, V_\mp^{(2)}] \\
[E_\pm^{(2)}, V_\pm^{(1)}] = 0 = [E_\pm^{(1)}, V_\pm^{(2)}] & [E_\pm^{(2)}, V_\mp^{(1)}] = \pm F_\mp^{(2)} = [E_\pm^{(1)}, V_\mp^{(2)}] \\
[U_\pm^{(2)}, U_\pm^{(1)}] = 0 = [V_\pm^{(2)}, V_\pm^{(1)}] & [V_+^{(2)}, V_-^{(1)}] = E_3^{(n)} + F_3^{(2)} = [V_+^{(1)}, V_-^{(2)}] \\
[E_\alpha^{(2)}, E_\alpha^{(1)}] = 0 = [F_\alpha^{(2)}, F_\alpha^{(1)}] & [U_+^{(2)}, U_-^{(1)}] = E_3^{(n)} - F_3^{(2)} = [U_+^{(1)}, U_-^{(2)}] \\
[E_\pm^{(2)}, F_\pm^{(1)}] = \mp V_\pm^{(2)} = [E_\pm^{(1)}, F_\pm^{(2)}] & [E_\pm^{(2)}, F_\mp^{(1)}] = \pm U_\pm^{(2)} = [E_\pm^{(1)}, F_\mp^{(2)}] \\
[F_\pm^{(2)}, V_\mp^{(1)}] = \mp E_\mp^{(2)} = [F_\pm^{(1)}, V_\mp^{(2)}] & [F_\pm^{(2)}, V_\pm^{(1)}] = 0 = [F_\pm^{(1)}, V_\pm^{(2)}] \\
[F_\pm^{(2)}, U_\pm^{(1)}] = \mp E_\pm^{(2)} = [F_\pm^{(1)}, U_\pm^{(2)}] & [F_\pm^{(2)}, U_\mp^{(1)}] = 0 = [F_\pm^{(1)}, U_\mp^{(2)}] \\
[F_3^{(2)}, U_\pm^{(1)}] = \mp U_\pm^{(2)} = [F_3^{(1)}, U_\pm^{(2)}] & [F_3^{(2)}, V_\pm^{(1)}] = \pm V_\pm^{(2)} = [F_3^{(1)}, V_\pm^{(2)}] \\
[F_3^{(2)}, F_\pm^{(1)}] = \pm F_\pm^{(2)} = [F_3^{(1)}, F_\pm^{(2)}] & [F_3^{(2)}, E_\pm^{(1)}] = 0 = [F_3^{(1)}, E_\pm^{(2)}] \\
[E_+^{(2)}, E_-^{(1)}] = E_3^{(2)} = [E_+^{(1)}, E_-^{(2)}] & [F_+^{(2)}, F_-^{(1)}] = F_3^{(2)} = [F_+^{(1)}, E_-^{(2)}].
\end{array} \tag{31}$$

However, equation (15) is changed as follows:

$$[E_3^{(2)}, F_3^{(2)}] = \tfrac{1}{4}(U_-^{(1)} E_+^{(1)} F_-^{(1)} - F_+^{(1)} E_-^{(1)} U_+^{(1)} + V_-^{(1)} E_+^{(1)} F_+^{(1)} - F_-^{(1)} E_-^{(1)} V_+^{(1)}). \tag{32}$$

All relations other than equations (10) and (11) can also be generated on the basis of equation (32) by using Jacobi identities together with equations (30) and (31). This is the definition of $Y(so(5))$.

Substituting equations (23) into (22), by extensive calculation, equation (22) is changed into the following algebraic relations:

$$
\begin{array}{l}
[E_3^{(n)}, U_\pm^{(1)}] = \pm U_\pm^{(n)} = [E_3^{(1)}, U_\pm^{(n)}] \qquad [E_3^{(n)}, E_\pm^{(1)}] = \pm E_\pm^{(n)} = [E_3^{(1)}, E_\pm^{(n)}] \\
[E_3^{(n)}, V_\pm^{(1)}] = \pm V_\pm^{(n)} = [E_3^{(1)}, V_\pm^{(n)}] \qquad [E_3^{(n)}, F_\alpha^{(1)}] = 0 = [E_3^{(1)}, F_\alpha^{(n)}] \\
[E_\pm^{(n)}, U_\mp^{(1)}] = \mp F_\pm^{(n)} = [E_\pm^{(1)}, U_\mp^{(n)}] \qquad [E_\pm^{(n)}, U_\pm^{(1)}] = 0 = [E_\pm^{(1)}, U_\pm^{(n)}] \\
[U_\pm^{(n)}, V_\pm^{(1)}] = 0 = [U_\pm^{(1)}, V_\pm^{(n)}] \qquad [U_\pm^{(n)}, V_\mp^{(1)}] = 0 = [U_\pm^{(1)}, V_\mp^{(n)}] \\
[E_\pm^{(n)}, V_\pm^{(1)}] = 0 = [E_\pm^{(1)}, V_\pm^{(n)}] \qquad [E_\pm^{(n)}, V_\mp^{(1)}] = \pm F_\mp^{(n)} = [E_\pm^{(1)}, V_\mp^{(n)}] \\
[U_\pm^{(n)}, U_\pm^{(1)}] = 0 = [V_\pm^{(n)}, V_\pm^{(1)}] \qquad [V_+^{(n)}, V_-^{(1)}] = E_3^{(n)} + F_3^{(n)} = [V_+^{(1)}, V_-^{(n)}] \\
[E_\alpha^{(n)}, E_\alpha^{(1)}] = 0 = [F_\alpha^{(n)}, F_\alpha^{(1)}] \qquad [U_+^{(n)}, U_-^{(1)}] = E_3^{(n)} - F_3^{(n)} = [U_+^{(1)}, U_-^{(n)}] \\
[E_\pm^{(n)}, F_\pm^{(1)}] = \mp V_\pm^{(n)} = [E_\pm^{(1)}, F_\pm^{(n)}] \qquad [E_\pm^{(n)}, F_\mp^{(1)}] = \pm U_\pm^{(n)} = [E_\pm^{(1)}, F_\mp^{(n)}] \\
[F_\pm^{(n)}, V_\mp^{(1)}] = \mp E_\mp^{(n)} = [F_\pm^{(1)}, V_\mp^{(n)}] \qquad [F_\pm^{(n)}, V_\pm^{(1)}] = 0 = [F_\pm^{(1)}, V_\pm^{(n)}] \\
[F_\pm^{(n)}, U_\pm^{(1)}] = \mp E_\pm^{(n)} = [F_\pm^{(1)}, U_\pm^{(n)}] \qquad [F_\pm^{(n)}, U_\mp^{(1)}] = 0 = [F_\pm^{(1)}, U_\mp^{(n)}] \\
[F_3^{(n)}, U_\pm^{(1)}] = \mp U_\pm^{(n)} = [F_3^{(1)}, U_\pm^{(n)}] \qquad [F_3^{(n)}, V_\pm^{(1)}] = \pm V_\pm^{(n)} = [F_3^{(1)}, V_\pm^{(n)}] \\
[F_3^{(n)}, F_\pm^{(1)}] = \pm F_\pm^{(n)} = [F_3^{(1)}, F_\pm^{(n)}] \qquad [F_3^{(n)}, E_\pm^{(1)}] = 0 = [F_3^{(1)}, E_\pm^{(n)}] \\
[E_+^{(n)}, E_-^{(1)}] = E_3^{(n)} = [E_+^{(1)}, E_-^{(n)}] \qquad [F_+^{(n)}, F_-^{(1)}] = F_3^{(n)} = [F_+^{(1)}, E_-^{(n)}]
\end{array}
\tag{33}
$$

$$
\begin{array}{l}
[\tilde{E}_3^{(n)}, U_\pm^{(1)}] = \pm \tilde{U}_\pm^{(n)} = [E_3^{(1)}, \tilde{U}_\pm^{(n)}] \qquad [\tilde{E}_3^{(n)}, V_\pm^{(1)}] = \pm \tilde{V}_\pm^{(n)} = [E_3^{(1)}, \tilde{V}_\pm^{(n)}] \\
[\tilde{E}_3^{(n)}, E_\pm^{(1)}] = \pm \tilde{E}_\pm^{(n)} = [E_3^{(1)}, \tilde{E}_\pm^{(n)}] \qquad [\tilde{E}_3^{(n)}, F_\alpha^{(1)}] = 0 = [E_3^{(1)}, \tilde{F}_\alpha^{(n)}] \\
[\tilde{E}_\pm^{(n)}, U_\pm^{(1)}] = 0 = [E_\pm^{(1)}, \tilde{U}_\pm^{(n)}] \qquad [\tilde{E}_\pm^{(n)}, U_\mp^{(1)}] = \mp \tilde{F}_\pm^{(n)} = [E_\pm^{(1)}, \tilde{U}_\mp^{(n)}] \\
[\tilde{E}_\pm^{(n)}, V_\pm^{(1)}] = 0 = [E_\pm^{(1)}, \tilde{V}_\pm^{(n)}] \qquad [\tilde{E}_\pm^{(n)}, V_\mp^{(1)}] = \mp \tilde{F}_\mp^{(n)} = [E_\pm^{(1)}, \tilde{V}_\mp^{(n)}] \\
[\tilde{E}_+^{(n)}, E_-^{(1)}] = \tilde{E}_3^{(n)} - I_0^{(n)} = [E_+^{(1)}, \tilde{E}_-^{(n)}] \qquad [\tilde{F}_+^{(n)}, F_-^{(1)}] = \tilde{F}_3^{(n)} - I_0^{(n)} = [F_+^{(1)}, \tilde{F}_-^{(n)}] \\
[\tilde{E}_\pm^{(n)}, F_\pm^{(1)}] = \mp \tilde{V}_\pm^{(n)} = [\tilde{F}_\pm^{(n)}, E_\pm^{(1)}] \qquad [\tilde{E}_\pm^{(n)}, F_\mp^{(1)}] = \pm \tilde{U}_\pm^{(n)} = [E_\pm^{(1)}, \tilde{F}_\mp^{(n)}] \\
[\tilde{U}_+^{(n)}, U_-^{(1)}] = \tilde{E}_3^{(n)} - \tilde{F}_3^{(n)} = [U_+^{(1)}, \tilde{U}_-^{(n)}] \qquad [\tilde{V}_+^{(n)}, V_-^{(1)}] = \tilde{E}_3^{(n)} + \tilde{F}_3^{(n)} = [V_+^{(1)}, \tilde{V}_-^{(n)}] \\
[\tilde{U}_\pm^{(n)}, V_\mp^{(1)}] = \mp Y_\mp^{(n)} = [U_\pm^{(1)}, \tilde{V}_\mp^{(n)}] \qquad [\tilde{U}_\pm^{(n)}, V_\pm^{(1)}] = \mp X_\pm^{(n)} = [\tilde{V}_\pm^{(n)}, U_\pm^{(1)}] \\
[\tilde{F}_3^{(n)}, U_\pm^{(1)}] = \mp \tilde{U}_\pm^{(n)} = [F_3^{(1)}, \tilde{U}_\pm^{(n)}] \qquad [\tilde{F}_3^{(n)}, V_\pm^{(1)}] = \mp \tilde{V}_\pm^{(n)} = [\tilde{V}_\pm^{(n)}, F_3^{(1)}] \\
[\tilde{F}_3^{(n)}, F_\pm^{(1)}] = \pm \tilde{F}_\pm^{(n)} = [F_3^{(1)}, \tilde{F}_\pm^{(n)}] \qquad [\tilde{F}_3^{(n)}, E_\alpha^{(1)}] = 0 = [F_3^{(1)}, \tilde{E}_\alpha^{(n)}] \\
[\tilde{F}_\pm^{(n)}, U_\mp^{(1)}] = 0 = [F_3^{(1)}, \tilde{U}_\mp^{(n)}] \qquad [\tilde{F}_\pm^{(n)}, U_\pm^{(1)}] = \mp \tilde{E}_\pm^{(n)} = [F_\pm^{(1)}, \tilde{U}_\pm^{(n)}] \\
[\tilde{F}_\pm^{(n)}, V_\pm^{(1)}] = 0 = [F_\pm^{(1)}, \tilde{V}_\pm^{(n)}] \qquad [\tilde{F}_\pm^{(n)}, V_\mp^{(1)}] = \pm \tilde{E}_\mp^{(n)} = [\tilde{V}_\mp^{(n)}, F_\pm^{(1)}] \\
[\tilde{E}_3^{(n)}, E_3^{(1)}] = 0 = [F_3^{(1)}, \tilde{F}_3^{(n)}] \qquad [\tilde{E}_\pm^{(n)}, E_\pm^{(1)}] = \mp X_\pm^{(n)} \qquad [\tilde{F}_\pm^{(n)}, F_\pm^{(1)}] = \mp Y_\pm^{(n)} \\
[\tilde{U}_\pm^{(n)}, U_\pm^{(1)}] = 0 \qquad [\tilde{V}_\pm^{(n)}, V_\pm^{(1)}] = 0 \qquad [X_\pm^{(n)}, E_\mp^{(1)}] = \mp 2\tilde{E}_\pm^{(n)} \qquad [X_\pm^{(n)}, E_\pm^{(1)}] = 0 \\
[X_\pm^{(n)}, U_\mp^{(1)}] = \mp 2\tilde{V}_\pm^{(n)} \qquad [X_\pm^{(n)}, U_\pm^{(1)}] = 0 \\
[X_\pm^{(n)}, V_\mp^{(1)}] = \mp 2\tilde{U}_\pm^{(n)} \qquad [X_\pm^{(n)}, V_\pm^{(1)}] = 0 \\
[X_\pm^{(n)}, E_3^{(1)}] = \mp 2X_\pm^{(n)} \qquad [X_\pm^{(n)}, F_3^{(1)}] = 0 \qquad [X_\pm^{(n)}, F_\pm^{(1)}] = 0 \qquad [X_\pm^{(n)}, F_\mp^{(1)}] = 0 \\
[Y_\pm^{(n)}, E_3^{(1)}] = 0 \qquad [Y_\pm^{(n)}, F_3^{(1)}] = \mp 2Y_\pm^{(n)} \qquad [Y_\pm^{(n)}, E_\pm^{(1)}] = 0 \qquad [Y_\pm^{(n)}, E_\mp^{(1)}] = 0 \\
[Y_\pm^{(n)}, U_\pm^{(1)}] = \mp 2\tilde{V}_\pm^{(n)} \qquad [Y_\pm^{(n)}, U_\mp^{(1)}] = 0 \qquad [Y_\pm^{(n)}, F_\pm^{(1)}] = 0 \\
[Y_\pm^{(n)}, F_\mp^{(1)}] = \mp 2\tilde{F}_\pm^{(n)} \qquad [Y_\pm^{(n)}, V_\pm^{(1)}] = 0 \qquad [Y_\pm^{(n)}, V_\mp^{(1)}] = \pm 2\tilde{U}_\mp^{(n)} \\
[I_0^{(n)}, E_\pm^{(1)}] = \mp 2\tilde{E}_\pm^{(n)} \qquad [I_0^{(n)}, F_\pm^{(1)}] = \mp 2\tilde{F}_\pm^{(n)} \\
[I_0^{(n)}, U_\pm^{(1)}] = 0 \qquad [I_0^{(n)}, V_\pm^{(1)}] = 0 \qquad [I_0^{(n)}, E_3^{(1)}] = 0 \qquad [I_0^{(n)}, F_3^{(1)}] = 0 \\
[\tilde{E}_\alpha^{(n)}, I_0^{(1)}] = 0 \qquad [\tilde{F}_\alpha^{(n)}, I_0^{(1)}] = 0 \qquad [\tilde{U}_\pm^{(n)}, I_0^{(1)}] = 0 \qquad [\tilde{V}_\pm^{(n)}, I_0^{(1)}] = 0 \\
[X_\pm^{(n)}, I_0^{(1)}] = 0 \qquad [Y_\pm^{(n)}, I_0^{(1)}] = 0 \qquad [I_0^{(n)}, I_0^{(1)}] = 0 \qquad [E_\alpha^{(n)}, I_0^{(1)}] = 0 \\
[F_\alpha^{(n)}, I_0^{(1)}] = 0 \qquad [U_\pm^{(n)}, I_0^{(1)}] = 0 \qquad [V_\pm^{(n)}, I_0^{(1)}] = 0 \qquad (\alpha = \pm, 3).
\end{array}
\tag{34}
$$

From equations (34) we may establish that $I_0^{(n)}$ is the Casimor operator.

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